NUMERICAL EXTRACTION OF DISPERSION CURVES
USED IN LAMB WAVE INSPECTIONS

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Lamb wave method is a long range inspection technique which is considered to have a nice future in
the field of structural health monitoring. In recent years, the applications are expanding towards
composite structures. One of the main problems faced when using the Lamb wave method is the most
appropriate tuning of the frequency of generated waves, in order to have the adequate transmission,
able to properly propagate in the material, interfere with defects/damages and to be received in good
conditions. In order to get the optimum generation, extensive simulation has to be done. Knowledge
of dispersion curves is essential in this stage, and in the subsequent experimental stage. The present
work presents an original, fast and reliable method enabling the researcher to get all the range of
dispersion curves using only numerical simulation.

Keywords: Lamb waves; Dispersion curves; Finite element method.

1. INTRODUCTION

Conventional ultrasonic methods, such as pulse-echo, have been used successfully to interrogate
structural components. However, the application of these traditional techniques has been limited to testing
relatively simple geometries or interrogating the region in the immediate vicinity of the transducer. A new,
more promising ultrasonic methodology uses guided waves to examine structural components. The
advantages of this technique include: its ability to test the entire structure in a single measurement; and its
capacity to test inaccessible regions of complex components.

The plane waves travelling in a free isotropic plate case are a combination of longitudinal and
vertically polarized shear bulk waves propagating in a direction parallel to the plate. These two types of
waves are coupled by the reflections in the plate. The resulting guided waves travelling along the plate axis
in an isotropic material are called Lamb waves, which may have symmetric and antisymmetric modes.
Similar and supplementary modes exist also in tubes.

The increasing use of advanced composites in structural components, such as aerospace structures,
marine vehicles, automotive parts, and many other applications has led to extensive research activities in the
field of composite materials. The most widely used composite structures are laminates consisting of fibre-
reinforced laminae that are bonded together to achieve better structural performance than conventional
materials. To understand the dynamic response to use in damage detection, the theory of elastic waves in
composite laminates, particularly the dispersion curves, is desirable to be developed. However, due to
anisotropic of laminates, the exact analytical treatment of waves in composites is much more complicated
and consumes more computational cost than that of waves in isotropic plates and tubes.

Perhaps the most significant consequence of elastic anisotropy is the loss of pure wave modes for
general propagation directions. This fact also implies that the direction of group velocity, i.e., energy flow,
does not generally coincide with the wave vector or wave front normal. In addition, the distinction between
wave mode types in generally anisotropic plates is somehow artificial, since the equations for classical Lamb
and shear horizontal modes and symmetric and anti-symmetric modes generally are coupled.

Tang et al. [1] investigated the flexural wave motion in symmetric cross-ply and quasi-isotropic
laminates by both elasticity theory and first-order plate theory, and then compared the analytical results with
Numerical extraction of dispersion curves used in lamb wave inspections

One-layered anisotropic media, i.e., composite lamina. Nayfeh [3] developed a transfer matrix technique to obtain the relation for dispersion curves of elastic waves propagating in multilayered anisotropic media, i.e., composite laminate. Yuan and Hsieh [4] obtained the exact solutions of elastic waves in composite cylindrical shells based on 3-D anisotropic elasticity theory. However, these studies only obtained the dispersion relations for phase velocity, but not extended to group velocity.

Approximate plate theories have been extremely useful in providing analytical solutions to a variety of problems involving static and dynamic response of laminated plates and tubes of finite dimensions. Furthermore, for computing the dispersion relations of elastic waves in composites, approximate plate theories can offer much higher computational efficiency than 3-D elasticity theory since approximate plate theories avoid solving time-consuming transcendental equations. This feature of approximate plate theories is very useful to achieve an on-line structural health monitoring system in practice. The laminated plate theories are direct extensions of those developed earlier for homogeneous isotropic and orthotropic plates, where use is made of displacement fields which do not account for interface continuity explicitly between the lamina. The existing higher-order plate theory still needs to be extended to analyze elastic waves propagating along arbitrary direction in composites.

Concerning wave propagation in symmetric laminates, there are two types of uncoupled waves: symmetric (extensional and/or shear horizontal) waves and anti-symmetric (flexural and/or shear horizontal) waves. For non-symmetric laminates, the motion is more complicated and it is difficult to be analysed. In general, a guided wave consists of many different modes that propagate independently through the structure. The measured wave propagation quantity (waveform) is a superposition of all these modes. These waves interact with defects, and with geometrical features such as corners and curved surfaces, causing reflections and mode conversion. Definitely, propagation of guided waves in a complex structure is a complicated phenomenon that is difficult to understand and interpret.

One approach towards modelling guided wave propagation phenomena is to analytically solve the governing differential equations of motion under associated boundary conditions. This procedure can be done for specimens with simple geometries and without defects. Unfortunately, these equations become intractable for members with complicated geometries and damages/defects. Another approach to this problem is a numerical solution; the main advantage being the overcome of difficulties associated with complicated geometries and defects. There are basically two numerical methods which can be used for this problem: the finite element method (FEM) and the boundary element method (BEM). The primary advantage of FEM is that a good number of commercial FE codes is now available, with user friendly menus and providing sophisticated pre- and post-processing options. In addition, previous researchers [5] used the FEM to numerically calculate dispersion curves in a plate, clearly demonstrating that it is possible to use this method for guided wave propagation problems. In order to get dispersion curves out of finite element data, obtained by direct integration of equation of motion, Moser [6] used a 2D Fast Fourier Transformation but this method is not very accurate and imply a large effort of computation.

The objective of this work is to compare known, analytical, solutions of guided wave propagation problems with numerically obtained solutions, in order to establish the validity of FEM for modelling plate and annular wave guides. To this end, the paper presents modelling fundamentals concerning wave propagation in almost every type of structure - isotropic, orthotropic, anisotropic - by using numerical approach. It is important to note that this work uses a commercial, general-purpose FE code, namely ANSYS 10, the research version, with the formerly presented advantages. However, such general-purpose FE code lacks certain features, such as energy absorbing elements, that are available in specialized wave propagation codes. As a result, a secondary aim of this work is to find out the accuracy that can be reached with a non-specialized, general-purpose FE code.

2. GENERAL ASPECTS OF WAVES

In an arbitrary defined medium the are two fundamental types of waves: a longitudinal and transversal one. If the medium is finite, reflection phenomena there arise at the borders. If the medium is made from different materials, wave refraction phenomena will be added. In a finite plate or tube, longitudinal and transversal waves are successively reflected and collected, generating a propagating wave of a particular
shape. The waves are generated by adequate devices and can be described by a sum of harmonics. Using a controlled excitation (in time and space) it is possible to obtain a pure harmonic wave.

In a plate or a tube, several harmonic waves may exist at a given frequency, simply named modes. As a general rule, the number of modes increases with frequency. Function of spatial configuration, these modes have different names. In practice, only particular modes are used, easy to generate and measure.

A one dimensional progressive wave can be analytically described in time by

\[ \Psi(x,t) = A \cos(\omega t - kx), \]  

where \( A \) is the amplitude, \( \omega \) is the circular frequency, \( t \) is time, and \( k \) the wave number.

The following relationships link the previously defined quantities:

\[ \omega = 2\pi f = \frac{2\pi}{T}; \quad k = \frac{2\pi}{\lambda}, \]

where \( f \) is the wave frequency, \( T \) is the wave period and \( \lambda \) is the wave length.

The relationship between \( \omega \) and \( k \) is nonlinear and is named relation of dispersion. For a given wave and mode, having the length wave \( \lambda \) and a period \( T \), the phase velocity is defined by

\[ c_f = \frac{\lambda}{T} = \frac{\omega}{k}. \]

Group velocity is associated with the propagation velocity of a group of waves with similar frequency and is obtained from

\[ c_g = \frac{d\omega}{dk} \]

Like the phase velocity, group velocity is function of frequency and the considered wave mode.

Graphical representation of the quantities which define the waves (wave length, wave number, phase velocity and group velocity) function of the frequencies are denoted below as dispersion curves. These curves are used in practice to establish the work frequency and the wave mode to use in the propagation analysis.

For isotropic plates and tubes and also for some orthotropic layered ones, some researchers have developed dedicated codes to plot the dispersion curves, like the code Disperse, developed at Imperial College of London, by B. Pavlakovic & M. Lowe (www.ndt.imperial.ac.uk). However, for tubes, where the type of modes are much greater than in the plates, these codes may be unsuccessful. Moreover, for noncircular tubes or for complicated layered composite structures these codes can not be used.

A very simple and efficient method to obtain the dispersion curves, in a certain frequency range and for any usual structure which can be modelled using FEM, is presented below. Essentially, this method consists in a series of several modal analysis of a representative part of the analysed structure. Thus, for different wave lengths, one can find the mode shapes and corresponding natural frequencies by solving some eigenvalue problems. Then, using Eqs. (2) - (4), the dispersion curve may be finally plotted.

3. APPLICATION OF THE FINITE ELEMENT METHOD

This study is based on the assumptions of linear elasticity. The general equations of motion in matrix form are given as [8]:

\[ [M][\ddot{u}] + [C][\dot{u}] + [K][u] = \{F\}, \]  

where \([M]\) is the structural mass matrix; \([C]\) is the structural damping matrix; \([K]\) is the structural stiffness matrix; \(\{F\}\) is the vector of applied loads; and \(\{u\}, \{\dot{u}\}, \{\ddot{u}\}\) are the displacement vector and its time derivatives, respectively. General equation (5) can be solved using the direct integration techniques and can capture the travelling wave in a structure [7].

If damping and the applied forces are neglected in a structural analysis, the equation of motion (5) is transformed in an equation describing the free vibrations:
\[ [M] \ddot{u} + [K] u = \{0\}, \quad (6) \]

where, in the general case, the geometric stiffness matrix \([K_g]\), due to the initial stress distribution in the structure, is included in the stiffness matrix of the structure \([K]\). The mass and stiffness matrices are symmetric and constant. At this stage, one considers the boundary conditions imposed in equation (6). The solution of this equation is in the form

\[ \{u\} = \{\phi\} e^{i\omega t}, \quad (7) \]

where \(\{\phi\}\) is the mode shape, independent of time, and \(\omega\) is the eigen circular frequency. Including the solution and its second derivate into Eq. (6) yields

\[ (-\omega^2 [M] + [K]) \{\phi\} = \{0\}. \quad (8) \]

This generalized eigenvalue problem has \(n\) pairs of eigenvalues \(\omega_j^2\) and associated eigenvectors \(\{\phi_j\}\). Usually, the eigenvectors are normalized with respect to the mass matrix, thus obtaining

\[ \{\phi_j\}^T [M] \{\phi_j\} = 1; \quad j = 1, \ldots, n. \quad (9) \]

The eigenvectors are orthogonal with respect to the mass and stiffness matrices, and they are arranged in the ascending order of the eigenvalues.

Beside the mode shape, the usual parameter that is used by the engineers is the eigenfrequency:

\[ f_j = \frac{\omega_j}{2\pi}. \quad (10) \]

The smallest eigenfrequency is the fundamental one. If the structure has mechanism or rigid body motion, one obtains null eigenfrequencies corresponding to these movements. If the structure has symmetry, coincident eigenfrequencies may be obtained.

The mode shapes obtained from the modal analysis, by solving Eq. (8) may be particular Lamb modes, and the corresponding frequencies (10) are the Lamb mode frequencies if Eq. (8) compels with the boundary conditions of Lamb modes. So, a FE model of a representative portion from a plate or a tube must be forced to describe the Lamb mode. This can be done using some restrictions applied to a number of nodes. The next paragraph presents the mathematical aspects of such condition concerning the assembled structural matrices.

4. THE SOLUTION OF A LINEAR SYSTEM WITH RESTRICTIONS

The conditions imposed to displacements and rotations in certain nodes can be mathematically interpreted as restrictions associated to a system of equations. Such conditions, in the form of imposed displacements or cinematic relations between certain degrees of freedom (d.o.f.s), are sometimes called coupling relations between the nodal parameters [9].

There are many mathematical procedures for obtaining the solution of a system that contains restrictions: elimination of a number of equations equal to the number of restrictions, the method of Lagrange multipliers and the method of penalty functions, to mention just a few.

Physically, a restriction can include a single d.o.f., for example a constraint or an imposed value for a nodal displacement on a certain direction, or two or more d.o.f.s, for example initially unknown imposed displacements on two degrees of freedom. The restrictions imposed on two or more d.o.f.s are usually considered when rigid elements or mechanism movements are present.

To simplify of the presentation, let us consider only the static equilibrium part from equation (5), written in the form

\[ [K] U = \{F\}, \quad (11) \]

that should be solved with the linearly independent restrictions written as

\[ [G] U = \{Q\}. \quad (12) \]
The matrix \( G \) is rectangular, with constant coefficients, and the number of lines is equal to the number of restrictions. The vector \( Q \) has also constant terms that, in most practical cases, are null. Some methods for including the restrictions in the equilibrium equation of the model are further presented.

**The elimination method.** The vector of nodal displacements in Eq. (11) can be written in the form:

\[
\{ U \} = \left\{ \{ U_r \} \{ U_e \} \right\}^T,
\]

where \( \{ U_r \} \) are the \( r \) constrained displacements and \( \{ U_e \} \) – the \( e \) displacements that are to be eliminated (note that \( n = r + e \)). Now, equation (12) can be written in the form

\[
\begin{bmatrix} G_r \\ G_e \end{bmatrix} \left[ \begin{bmatrix} \{ U_r \} \\ \{ U_e \} \end{bmatrix} \right] = \{ 0 \}.
\]

(14)

Because the number \( r \) of linearly independent equations is smaller than the number \( n \) of equilibrium equations, it results that the matrix \( G_e \) is not singular. From (14), it yields:

\[
\{ U_e \} = -[G_e]^{-1}[G_r]\{ U_r \},
\]

(15)

relation that can be included in the transformation:

\[
\left\{ \begin{bmatrix} \{ U_r \} \\ \{ U_e \} \end{bmatrix} \right\} = \left[ \begin{bmatrix} I_r \\ -[G_e]^{-1}[G_r] \end{bmatrix} \right]\{ U_r \}.
\]

(16)

One can rewrite (16) in the form:

\[
\{ U \} = [T] \{ U_r \}; \quad \text{with} \quad [T] = \left[ \begin{bmatrix} I_r \\ -[G_e]^{-1}[G_r] \end{bmatrix} \right],
\]

(17)

where \([I] \) is the unity matrix.

If expression (17) is replaced in (11) and left multiplied with \([T]^T \), a system of \( r \) equations is obtained in the form:

\[
[K_r][U_r] = \{ F_r \},
\]

(18)

in which

\[
[K_r] = [T]^T[K][T]; \quad \{ F_r \} = [T]^T\{ F \}.
\]

(19)

If Eq. (6) is considered instead of Eq. (11) for the boundary condition not yet imposed, one gets

\[
[M_r][\dot{u}_r] + [K_r][u_r] = \{ 0 \},
\]

(20)

where the reduced mass and stiffness matrices are

\[
[M_r] = [T]^T[M][T]; \quad [K_r] = [T]^T[K][T].
\]

(21)

Solving Eq. (20), results the \( r \) set displacements, the \( e \) set displacements can be recovered from (15).

**The method of Lagrange multipliers.** In this method, a function of non-linear variables is minimized. In the case of structural analysis, one uses the expression of the potential energy, written in matrix form as

\[
\Pi = \frac{1}{2}\{ U \}^T[K]\{ U \} - \{ U \}^T\{ F \}.
\]

(22)

If equation (12) multiplied with the vector \( \{ \sigma \}^T \) is added to the potential, the following expression of modified potential energy function is obtained:

\[
\Pi = \frac{1}{2}\{ U \}^T[K]\{ U \} - \{ U \}^T\{ F \} + \{ \sigma \}^T\left( [G][U] - \{ Q \} \right)
\]

(23)

The vector \( \{ \sigma \}^T \) represents the Lagrange multipliers, that are forces keeping the equilibrium of the structure.
If the following extreme conditions are imposed
\[
\left\{ \frac{\partial \Pi}{\partial [U]} \right\} = \{0\}; \quad \left\{ \frac{\partial \Pi}{\partial [\sigma]} \right\} = \{0\},
\]
(24)
a system of \( n + r \) equations is obtained in the form:
\[
\begin{bmatrix}
K & G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
[U] \\
[\sigma]
\end{bmatrix}
= \begin{bmatrix}
F \\
Q
\end{bmatrix},
\]
(25)

If, again, Eq. (6) is considered instead of Eq. (11) for the boundary condition not yet imposed, in the
same way, with little corrections due to the necessity to include the kinetic energy and the Hamilton’s
principle, the following equation results
\[
\begin{bmatrix}
M & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{\sigma\}
\end{bmatrix}
+ \begin{bmatrix}
K & G^T \\
G & 0
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{\sigma\}
\end{bmatrix}
= \begin{bmatrix}
\{0\} \\
\{0\}
\end{bmatrix},
\]
(26)

The method of penalty function. In this method, the unknowns are approximately obtained, and the
restrictions are approximately fulfilled. Eq. (12) is rewritten in the form
\[
\{r\} = [G][U] - \{Q\},
\]
(27)
and is introduced in the Eq. (22), thus yielding:
\[
\Pi = \frac{1}{2} \{U\}^T [K] \{U\} - \{U\}^T \{F\} + \frac{1}{2} \{r\}^T [\alpha] \{r\},
\]
(28)
The supplementary term \( \{r\}^T [\alpha] \{r\} / 2 \) is called penalty function with matrix \([\alpha]\) in diagonal form. If
Eq. (27) is introduced in Eq. (28) and the condition of minimum potential
\[
\left\{ \frac{\partial \Pi}{\partial [U]} \right\} = \{0\}
\]
(29)
is imposed, one obtains
\[
\]
(30)
The matrix \([G]^T [\alpha] [G]\), called penalty matrix, is added to the stiffness matrix of the structure, and the
vector \([G]^T [\alpha] \{Q\} \) is added to the vector of the initial nodal loads. If \([\alpha] = \{0\}\), then the applied restrictions
can be neglected. If the norm of matrix \([\alpha]\) increases, the vector of nodal displacements \([U]\) is modified to
to better satisfy the restrictions. It is convenient to have non-dimensional terms in matrix \([\alpha]\).

For modal analysis, relation (6), can be rewrite
\[
\]
(32)
In this method, the initial dimension of the problem remains unaltered. The matrix \([\alpha]\) must have terms
much greater than the stiffness values corresponding to the main diagonal of matrix \([K]\). For this reason,
numerical problems may appear due to an important increase of the conditioning parameter of the modified
stiffness matrix \([K] + [G]^T [\alpha] [G]\).

Coupling d.o.f.s into a set makes that results calculated for one member of the set are the same for all
members of the set. Generally, the coupling procedure can be used to model various joint and hinge effects,
but in this work is used to enforce the obtained Lamb mode shape. A more general form of coupling can be
done with constraint equations. For analysis, a list of nodes is defined along with the nodal directions in
which these nodes are to be coupled. As a result of this coupling, these nodes are forced to have the same
displacement in the specified nodal coordinate direction. The amount of the displacement is unknown until
the analysis is completed. A set of coupled nodes which are not coincident, or which are not along the line of
the coupled displacement direction, may produce a force or an applied moment which represents the force necessary to assure the desired displacement.

5. DETAILED METHODOLOGY AND EXAMPLES

In order to understand the behaviour and robustness of FEM applied to the solution of guided wave problems, a relatively simple geometry is considered first: a 4 mm thick aluminium plate, with Young’s modulus $E = 70$ GPa, Poisson’s ratio $\nu = 0.33$ and mass density $\rho = 2700$ kg/m$^3$. This geometry has the advantage of a known analytical solution - the Rayleigh–Lamb equation [10].

This application uses two-dimensional (2D) solid structural elements that model plane strain in the $Z$ direction. These elements are defined by four nodes with 2 d.o.f.s at each node: translations in the $X$ and $Y$ directions (element type PLANE42). The mass distribution is uniform for all elements. The element size is chosen so that the propagating waves are spatially identified for a frequency range of interest until 700 kHz. In [6], it is recommended to use more than 10-20 nodes per wavelength, which can be expressed as:

$$l_e \leq \frac{\lambda_{\min}}{10-20}$$

where $l_e$ is the element length and $\lambda_{\min}$ is the shortest wavelength of interest. If highly accurate numerical results are needed, Eq. (33) might not be sufficient, and a finer meshing might be required.

Considering [10], it was written a MATLAB function to get and plot the dispersion curves of isotropic plates. To solve the transcendental Rayleigh-Lamb equations, the MATLAB function `fzero` was used. The results obtained with this user written function was confirmed by results from [10] and this function was used to compare the results obtained by FEM.

The FE model proper to be used for this application consist in a rectangle, having the base $L$ and the height $h = 4$ mm. At the start, $L$ can be almost arbitrarily chosen in the range (10-20) $h$, for example 60 mm. The rectangle was uniformly meshed with 6000 elements and results 6321 nodes.

In order to obtain the Lamb modes, each pair of the symmetric nodes from the two ends of the model (Fig. 1) was coupled. For example, considering the symmetric nodes $n_l$ and $n_r$ we have the horizontal and vertical displacements coupled: $UX(n_l) = UX(n_r)$ and $UY(n_l) = UY(n_r)$. With these restrictions, the mode shapes obtained from a modal analysis (Fig. 2) correspond to Lamb modes.

It can be observed that for an initial model of length $L$, one obtains the Lamb modes with the wave lengths

$$\lambda = \frac{L}{p}; \quad p = 1,2,3,...,$$

For different wave lengths, one can use a different initial length $L$. If the frequencies corresponding to the obtained Lamb modes are captured (Table 1), using the relation (2) and (3) it is trivial to find the wave number and the phase velocity. The group velocity must be obtained by numerical derivation, using (4). Since in practice the wave frequencies are obtained only for a limited range of wave lengths $\lambda$, this derivative can not be accurately obtained. To overcome this lack, a supplementary analysis for a new length $L' = L \pm \Delta L$, with slight variation around the initial length $L$, must be performed. In this way, the errors of the derivative can be efficiently controlled. The value of $\Delta L$ may be for example 0.5÷2.5% of $L$, depending on the imposed accuracy. The variation $\Delta L$ must not be very small, because it is necessary to
obtain a shift in the frequencies in respect to the two sets of wave length, corresponding to $L$ and $L'$, knowing that the eigenvalue problem is based on an iterative algorithm. The results from Table 1 correspond to a “perturbation” of $L$ of 0.5%.

$$\lambda = L$$

$$\lambda = \frac{L}{2}$$

$$\lambda = \frac{L}{3}$$

<table>
<thead>
<tr>
<th>Mode type</th>
<th>$\lambda = L$</th>
<th>$\lambda = \frac{L}{2}$</th>
<th>$\lambda = \frac{L}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S0</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>S1</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>A0</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
<tr>
<td>A1</td>
<td>![Image]</td>
<td>![Image]</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

**Fig. 2: Lamb modes obtained using FEM**

Table 1: Frequencies of the Lamb modes for an aluminium plate of thickness 4 mm, obtained using FEM

<table>
<thead>
<tr>
<th>$\lambda$ [mm]</th>
<th>S0 $L = 60$ mm</th>
<th>S0 $L' = 60.3$ mm</th>
<th>A0 $L = 60$ mm</th>
<th>A0 $L' = 60.3$ mm</th>
<th>A1 $L = 60$ mm</th>
<th>A1 $L' = 60.3$ mm</th>
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<tbody>
<tr>
<td>60</td>
<td>89739</td>
<td>89294</td>
<td>10542</td>
<td>10441</td>
<td>402568</td>
<td>402453</td>
</tr>
<tr>
<td>30</td>
<td>178466</td>
<td>177592</td>
<td>38955</td>
<td>38605</td>
<td>435022</td>
<td>434620</td>
</tr>
<tr>
<td>20</td>
<td>264873</td>
<td>263609</td>
<td>78958</td>
<td>78303</td>
<td>481431</td>
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<td>345375</td>
<td>125463</td>
<td>124497</td>
<td>536228</td>
<td>535083</td>
</tr>
<tr>
<td>12</td>
<td>421488</td>
<td>419758</td>
<td>175395</td>
<td>174126</td>
<td>595689</td>
<td>594175</td>
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<tr>
<td>10</td>
<td>484304</td>
<td>482632</td>
<td>227061</td>
<td>225503</td>
<td>657319</td>
<td>655466</td>
</tr>
<tr>
<td>8.57</td>
<td>533003</td>
<td>531527</td>
<td>279549</td>
<td>277712</td>
<td>719266</td>
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<td>330262</td>
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<td>6.67</td>
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<td>630137</td>
<td>438093</td>
<td>435463</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Using the results from Table 1, the dispersion curves (points) were determined and plotted in Fig. 3. It can be noticed a very good correlation between the numerical results and the analytical ones.

For this application, all Lamb modes appear in pairs. The eigenvalue problem was solved using Lanczos algorithm, for the first 70 frequencies. This algorithm has the important advantage that it finds all the mode shapes in a given frequency range, so this algorithm can be used in a small number of steps (frequency range), to reduce the computation cost. The restrictions (couplings between nodes) were eliminated according to Eq. (21).

To obtain more points in dispersion curves, a new start length must be chosen.
Fig. 3: Dispersion curves of an aluminium plate (continuous lines: analytically obtained; circles, squares, triangles: numerically obtained using FEM)

Table 2: Group velocities [km/s] obtained using two different “perturbation”, 0.5% and 2.5% of the initial model (length $L=60$ mm).

<table>
<thead>
<tr>
<th>$\lambda$ [mm]</th>
<th>$\Delta L = 0.025L$</th>
<th>$\Delta L = 0.005L$</th>
<th>$\Delta L = 0.025L$</th>
<th>$\Delta L = 0.005L$</th>
<th>$\Delta L = 0.025L$</th>
<th>$\Delta L = 0.005L$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5.3667</td>
<td>5.3653</td>
<td>1.2218</td>
<td>1.2218</td>
<td>1.3869</td>
<td>1.3751</td>
</tr>
<tr>
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The next example consists in obtaining, by analytical approach, the dispersion curves for a steel tube (Young’s modulus $E = 212$ GPa, Poisson’s ratio $\nu = 0.29$ and mass density $\rho = 7850$ kg/m$^3$) with an outer diameter of 14 mm and a wall thickness 1 mm. In comparison with plates, in tubes may be more types of Lamb waves, presented in [11] for the chosen dimensions. For this example only three mode types are calculated, all presented in Fig. 4. The notation of these modes may be ambiguous, but in this example the symbols used for plates were kept for longitudinal modes.

The FE model uses two-dimensional axi-symmetric solid structural elements able to simulate harmonic modes. These elements are defined by four nodes with three d.o.f.s in each node: translations in the $X$, $Y$, and $Z$ directions (element type PLANE25). The chosen average element size is 0.05 mm, because the targeted frequency was around 3 MHz, for which Lamb modes of interest have a minimum wave length of 1 mm.

Like the previous example, the estimated frequencies for Lamb modes of interest were obtained for some initial lengths of the tube model ($L = 30$, 20, 10 and 6 mm) and are presented in the Table 3. The calculated group velocity, obtained for a perturbation of the initial lengths with 2.5% is presented in the Table 4 and the dispersion curves are given in Fig. 5. The numerical results are in good agreement with those obtained in [11].

![Fig. 4: Some Lamb modes in tubes](image-url)
Table 4: Group velocities obtained using a “perturbation” of 2.5% of the initial models length

<table>
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<tr>
<th>λ [mm]</th>
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<th>Longitudinal S0</th>
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<td>Group velocity [m/s]</td>
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Fig. 5: Dispersion curves of an steel tube (numerically obtained using FEA)
6. CONCLUSION

The results of this study clearly illustrate the effectiveness of using the FE method to model guided wave propagation problems and demonstrate the potential of the FEM for problems where analytical solution is not possible because of complicated component geometry. Dispersion curves in laminated composite plates can be modelled by both 3-D elasticity theory and a higher-order plate theory. The advantage of using 3-D elasticity theory is obvious: the “exact” solutions hold for all frequency ranges. However, the exact solutions consist of infinite wave modes, symmetric or anti-symmetric motions. Although 3-D elasticity solutions can provide an exact modelling capability, computational cost is too high as compared to a higher-order plate theory. For this reason, the latter can be used to obtain approximate solutions in composites.

In the analysis of wave propagation in composites, classic Kirchhoff plate theory is valid in the low frequency range or for wavelengths much greater than the plate thickness. A higher-order plate theory should be used in the high frequency range, where the wavelength is smaller. Accordingly, if the effects of transverse shear deformation and rotational inertia are taken into account, the classic plate theory evolves into first-order shear deformation plate theory, or so-called Mindlin plate theory. Compared with Mindlin plate theory, higher-order plate theory can represent the kinematics better and yield more accurate inter-lamina stress distributions. These conclusions may establish the adequate finite element type used for wave propagation, particularly for dispersion curve extraction.

The potential of proposed method is more important in the study of composite plates and tubes, even in the complex approach of shells or general cylindrical structures. Thus, it is possible to investigate the anisotropic plates, the prismatic tubes, the influence of initial stress in the structure, etc.

On another hand, this work is the first step needed to develop a structural health monitoring technique implying guided waves. It will be much easier to tune the experimental chain aimed to perform scanning with Lamb waves when the best frequencies to use are already set using the numerically obtained dispersion curves. This will add a lot in increasing efficiency of non-destructive inspection techniques.

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REFERENCES